ABSTRACT

In this study an analytical analysis has been performed on the Atkinson cycle to create a thermodynamic model that allows for the optimization of both net work output as well as cycle efficiency. The model is first based on the ideal Atkinson cycle; it is then expanded to include compression and expansion losses, followed by an exploration into variable specific heat effects by taking into account the specific heat of both cold air and burned gas. The result is explicit formulations of different operating pressure ratios that allow for optimization of net work output and efficiency, as well as showing how compression and expansion inefficiencies and more accurate use of specific heats affect cycle performance. The results obtained in this work are useful to understand how the net work output and efficiency are influenced by the above losses and working fluid properties.

Keywords: Atkinson cycle, wave disk engine, gas turbine, pressure-gain combustion, constant-volume combustion.

INTRODUCTION

Gas turbine engine performance can be significantly improved by implementing an unfamiliar thermodynamic cycle known as the Atkinson (aka Humphrey) cycle [1]. The Atkinson cycle is similar to the Brayton cycle but uses constant-volume combustion rather than constant-pressure combustion process. Compared to the Otto cycle, the Atkinson cycle benefits from full expansion of burned gas, which does not occur in the Otto cycle. Thermodynamic analysis of the Atkinson cycle shows that the implementation of this cycle significantly reduces entropy generation while producing more net work output, allowing for engine efficiency enhancement [2].
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Atkinson cycle. This hypothetical model assumes no losses, and constant specific heats of dry air. This model is the most simplistic, however it will be used as a guide to build other models based on it.

MODEL OF IDEAL AIR STANDARD ATKINSON CYCLE

The first model introduced is the ideal air standard model of the Atkinson cycle. This hypothetical model assumes no losses, and constant specific heats of dry air. This model is the most simplistic, however it will be used as a guide to build other models based on it.

Net work output for ideal air standard Atkinson cycle

To begin the analysis of the Atkinson cycle, the first law of thermodynamics is applied to the cycle [2]:

\[
\begin{align*}
\dot{w}_{\text{net}} &= c_p(T_3 - T_4) - c_p(T_2 - T_1) - v_2(P_3 - P_2) \\
&= c_p(T_3 - T_4) - c_p(T_2 - T_1) - R(T_3 - T_2) \\
&= c_v(T_3 - T_2) - c_p(T_4 - T_1)
\end{align*}
\]

(1)

The next step is to express Eq. (1) in terms of cycle temperature ratio, \(T_3/T_1\), and compressor pressure ratio, \(P_2/P_1\).

To start:

\[
\dot{w}_{\text{net}} = c_v T_2 \left( \frac{T_3}{T_2} - 1 \right) - c_p T_1 \left( \frac{T_4}{T_1} - 1 \right)
\]

(2)

Reference [8] has introduced the following expression using the definition of heat addition, \(q_{\text{in}} = c_v(T_3 - T_2)\), for the Atkinson cycle:

\[
\frac{T_4}{T_1} = \left( \frac{v_1}{v_2} \right)^{\frac{1-k}{k}} \left( \frac{q_{\text{in}}}{c_v T_1} + \frac{v_1}{v_2}^{k-1} \right)^{\frac{1}{k}} = \frac{T_3}{T_1} \left( \frac{T_3}{T_2} \right)^{\frac{1}{k}}
\]

(3)

By substituting Eq. (3) into Eq. (2), and normalizing \(\dot{w}_{\text{net}}\) by \(c_v T_1\), the following equation is obtained for the specific work:

A plot of Eq. (4) with pressure ratio as the independent variable is seen in Fig. 3 for several temperature ratios. There is a clear optimum pressure ratio for each temperature ratio that corresponds to maximum non-dimensional net work output. As temperature ratio increases, the optimum pressure ratio and corresponding (maximum) net work output increase as well. In addition, each temperature line decays to zero at very large pressure ratios. This implies that there exists a maximum pressure ratio for which there is zero net work output.

It is desired to derive an explicit expression for the optimum pressure ratio, \((r_{\text{opt}})_w\), at which the work output reaches a maximum. The optimum compression ratio value is obtained by differentiating the net work output in Eq. (4) with respect to \(r\) and equating it to zero for any given temperature ratio. This yields:

\[
(r_{\text{opt}})_w = t^{\frac{k}{k-1}}
\]

(5)

Substituting Eq. (5) into Eq. (4) gives:

\[
\left( \frac{\dot{w}_{\text{net}}}{c_v T_1} \right)_{\text{max}} = t - t^{\frac{1}{k+1}}(1 + k) + k
\]

(6)

Equation (6) calculates the maximum net work output of the ideal air standard Atkinson cycle for any given temperature ratio.

To solve for the maximum pressure ratio, \((r_{\text{max}})_w\) which results in zero net work output, Eq. (4) must be set equal to zero, and solved in terms of \(r\), yielding:

\[
(r_{\text{max}})_w = \left( t + k - k t^{\frac{1}{k+1}} (r_{\text{max}})_w^{\frac{1-k}{k}} \right)^{\frac{k}{k-1}}
\]

(7)

Equation (7) is a nonlinear equation in terms of \(r\), which may not be explicitly solved for \((r_{\text{max}})_w\). The solution can be

\[
\frac{\dot{w}_{\text{net}}}{c_v T_1} = t - r^{\frac{k-1}{k}} - k t^{\frac{1}{k+1}} r^{\frac{1-k}{k+1}} + k
\]

(4)

found by solving Eq. (7) numerically.
Efficiency for ideal air standard Atkinson cycle

Cycle efficiency for the Atkinson cycle is defined by:

$$\eta = \frac{W_{net}}{q_{in}} = \frac{W_{net}}{c_{v}(T_3 - T_2)}$$  \hspace{1cm} (8)

Combining Eq. (8) and Eq. (4) yields an equation for cycle efficiency in terms of $r$ and $t$ such that:

$$\eta = \frac{t - r^{k-1} - \frac{k}{k+1} \frac{1}{r^{k-1}} + \frac{k}{k+1}}{t - \frac{1}{r^{k-1}}}$$  \hspace{1cm} (9)

A plot of Eq. (9) for $t = 4, 5, 6$ is shown in Fig. 4 along with the corresponding Carnot efficiencies calculated by:

$$\eta_{carnot} = 1 - \frac{T_4}{T_3}$$  \hspace{1cm} (10)

As the compression ratio increases, the thermal efficiency increases, but the rate of increase diminishes at higher compression ratios. As expected, higher efficiencies are obtained as the operating temperature ratios rise. The maximum ideal efficiency will be the Carnot efficiency.

A discontinuity will occur at $t = r^{k-1}$ which implies that $T_2 = T_3$. Physically what causes this is due to the fact that at very large compressor pressure ratios, temperature $T_2$ approaches the temperature $T_3$, resulting in zero heat addition. For instance, for $t = 4$, the discontinuity occurs at $(r_{max})_\eta = 128$, for $t = 5$, $(r_{max})_\eta = 279.5$, and for $t = 6$, $(r_{max})_\eta = 529.1$. The corresponding maximum efficiencies are $\eta_{max} = 0.75, 0.8$, and $0.833$, respectively.

For each temperature ratio, there is a efficiency, $\eta_{(r_{opt})_{w}}$, such that the non-dimensional net work output is maximized. To obtain the corresponding thermal efficiency at maximum work output, Eq. (5) is substituted into Eq. (9). This yields:

$$\eta_{(r_{opt})_{w}} = 1 + \frac{k \left( 1 - \frac{1}{k+1} \right)}{t - \frac{1}{k+1}}$$  \hspace{1cm} (11)

Figure 4: Plot of ideal efficiency vs. compressor pressure ratio with corresponding Carnot efficiencies for $k=1.4$ and $t = 4, 5, 6$.

Equation (11) gives an explicit formula to find the efficiency corresponding to the maximum net work output for the ideal air standard Atkinson cycle at a given operating temperature ratio. This can be clearly seen in Fig. 5, which shows variations of non-dimensional net work output versus cycle efficiency for $t = 4, 5, 6$, with lines of constant compressor pressure ratio shown as a reference. These lines are graphical tools to find maximum point of net work output on each of the temperature curves. Such performance maps are useful for engine optimization.

Figure 5: Plot of ideal efficiency vs. non-dimensional net work output for $k =1.4$ and $t =4, 5, 6$ with lines of constant pressure ratio $(r = 2, 10, 20, 50)$.

AIR STANDARD ATKINSON CYCLE WITH COMPRESSION AND EXPANSION LOSSES

The next model explored is the Atkinson cycle with losses during compression and expansion processes (the so-called non-ideal cycle).
Net work output of air standard Atkinson cycle with compression and expansion losses

Compared to the ideal air standard Atkinson cycle, a more realistic model can be developed by applying isentropic efficiencies to both the compression ($\eta_c$) and expansion ($\eta_t$) processes. To begin deriving the net work output for the Atkinson cycle with compressor and turbine losses, Eq. (1) is considered again:

$$w_{net} = c_p(T_3 - T_4) - c_p(T_2 - T_1) - R(T_3 - T_2)$$

$$= c_pT_3 \left(1 - \frac{T_4}{T_3}\right) - c_pT_1 \left(\frac{T_2}{T_1} - 1\right) - RT_2 \left(\frac{T_3}{T_2} - 1\right)$$

(12)

From the definition of compression isentropic efficiency:

$$T_2 = \frac{r^{-1} - 1}{\eta_c} + 1$$

(13)

Likewise from the definition of expansion isentropic efficiency:

$$1 - \frac{T_4}{T_3} = \eta_t \left(1 - \frac{P_3}{P_4} \frac{r^{k-1}}{k}\right)$$

(14)

Before the losses can be integrated into Eq. (12), the right hand side of Eq. (14) must be written in terms of $r$ and $t$ only. This is obtained as follows:

$$\frac{P_4}{P_3} = \frac{P_4}{P_1} \frac{P_1}{P_2} \frac{P_2}{P_3} = \frac{P_1 T_2 T_1}{P_2 T_1 T_3} = r^{-1} t^{-1} \left(\frac{r^{k-1} - 1}{\eta_c} + 1\right)$$

(15)

Substituting Eq. (13), (14) and (15) into Eq. (12) and normalizing by $c_v T_1$ yields:

$$\frac{w_{net}}{c_v T_1} = \eta_t k t \left[1 - \left(\frac{k-1}{r^{-k} + 1}\right) - k \left(\frac{k-1}{r^{-k} + \eta_c}\right) + (1-k)\left(1 - \frac{k-1}{r^{-k} + \eta_c}\right)\right]$$

(16)

Equation (16) can easily be verified by setting $\eta_c = \eta_t = 1$, resulting in Eq. (4) which represents the ideal cycle.

Figure 6 compares both the ideal and non-ideal non-dimensional net work output with 90% losses during compression and expansion processes. The realistic net work output behaves similar to that of the ideal cycle. However, the net work output decreases significantly when losses are included. The curves with losses also have a steeper slope than the ideal curves. This implies that the pressure ratio, for which the net work output is maximized, $(r_{opt})_w$, will be lower when losses are included. To find this pressure ratio, the derivative of Eq. (16) is taken, and set equal to zero. This yields:

$$\frac{d}{dr} \left(\frac{w_{net}}{c_v T_1}\right)$$

$$= (-1 + k) \left(-\frac{r^{-k}}{r^{2}}\right)$$

$$\eta_t \left(r + k(-1 + \eta_c) r^1 \left(1 + \frac{-1 + r^{-1 + k}}{\eta_c} t^{1 - k}\right)\right)$$

$$/ (k\eta_c) = 0$$

(17)

Figure 7 shows the results of numerical methods applied to Eq. (17) to solve for $(r_{opt})_w$ at different temperature ratios, and for different loss values. For a given temperature ratio, the ideal cycle has the largest $(r_{opt})_w$. As losses accumulate, $(r_{opt})_w$ decreases, and the slope of the $(r_{opt})_w$ curve becomes shallower.

Finally, from Fig. 6, it is expected to find an $(r_{max})_w$ such that the net work output becomes zero. To obtain $(r_{max})_w$, Eq. (16) must be set equal to zero for a given temperature ratio, and the solved for $r$. Again, this solution must be numerically calculated.
Figure 7: Plot of $r_{\text{opt}}$ vs. temperature ratio for both ideal and non-ideal ($\eta_c = \eta_t = 0.85, 0.90, 0.95$) cycles, for $k = 1.4$.

Efficiency for air standard Atkinson cycle with compression and expansion losses

Continuing in the same fashion, an analysis of the cycle efficiency with losses is needed. To proceed, Eq. (16) is divided by $r^k$, and multiplied by $c_p T_1$:

$$\eta = \eta_t \frac{k t}{k - 1 - \frac{1}{\eta_c} \left[ \frac{k}{r} - 1 \right]} + \frac{(1 - k)}{\eta_c} \left[ t - \frac{k - 1}{r^k \eta_c} \right]$$

A plot of Eq. (18) is shown in Fig. 8 for $\eta_c = \eta_t = 0.9$, compared with the ideal efficiency. As seen, when losses are taken into account, there will be a maximum cycle efficiency for any fixed temperature ratio which is lower than the corresponding Carnot efficiency. The so-called optimum pressure ratio, $r_{\text{opt}}$, for a maximum cycle efficiency can be obtained when the derivative of Eq. (18) is taken, and set equal zero. Again explicit solution becomes difficult to obtain, and numerical methods are needed to solve for $r_{\text{opt}}$. Comparing the values of $(r_{\text{opt}})_{w}$ and $(r_{\text{opt}})_{\eta}$ from Figs. 6 and 8, respectively, it is concluded that for the optimum pressure ratio based on work occurs at a lower pressure ratio than the point of maximum efficiency at the same temperature ratio, i.e. $(r_{\text{opt}})_{w} > (r_{\text{opt}})_{\eta}$, similar to the Brayton cycle. Thus, an intermediate compression pressure ratio may be considered for performance optimization. Similarly, substituting $(r_{\text{opt}})_{\eta}$ into Eq. (18) would result in finding maximum efficiency for the Atkinson cycle with losses. Finally, to obtain maximum pressure ratio such that the efficiency will drop to zero, $(r_{\text{max}})_{\eta}$, can be obtained by setting Eq. (18) equal to zero and solving for $r$ numerically.

Similar to Fig. 5, Fig. 9 represents performance characteristics of the Atkinson cycle with losses included. Hence, Fig. 9 clearly shows maximum points for cycle both efficiency and specific net work output for given temperature ratios when losses are taken into account.

Figure 8: Plot of efficiency with losses and ideal vs. pressure ratio for $k = 1.4$, $t = 4, 5, 6$ and $\eta_c = \eta_t = 0.9$.

Figure 9: Plot of efficiency with losses vs. non-dimensional net work output for $k = 1.4$ and $t = 4, 5, 6$ with lines of constant pressure ratio ($r = 2, 5, 10, 20$) and $\eta_c = \eta_t = 0.9$. 

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PERFORMANCE OF IDEAL ATKINSON CYCLE WITH VARIABLE SPECIFIC HEATS

Another adjustment that can be made to the current thermodynamic model developed for the Atkinson cycle is to add rudimentary variable specific heat effects. The simplest way to see how variable specific heats could affect the performance of the cycle would be to assign a “cold” specific heat to states between points 1 and 2 of the cycle, and a “hot” specific heat to states between points 3 and 4 of the cycle.

Following the procedure described before, the net work output can be calculated by:

\[ w_{\text{net}} = c_{ph} (T_3 - T_4) - c_{pc} (T_2 - T_1) - v_k (P_3 - P_2) \]
\[ = c_{ph} (T_3 - T_4) - c_{pc} (T_2 - T_1) - [k_h T_3 - R_c T_2] \]
\[ = c_v T_2 \left( k_c \frac{T_2}{T_1} - 1 \right) + c_{vh} T_3 \left( 1 - k_h \frac{T_4}{T_3} \right) \]  

(19)

Normalizing by \( c_v T_1 \) yields:

\[ \frac{w_{\text{net}}}{c_v T_1} = \left( k_c - \frac{T_2}{T_1} \right) + \left( \frac{c_{vh} T_3}{c_v T_1} - \frac{c_{ph} T_4}{c_v T_1} \right) \]  

(20)

To keep the model consistent with discussions before, Eq. (20) should be written in terms of \( r \) and \( t \) alone. Thus:

\[ T_4 = T_3 \frac{T_4}{T_1} = T_3 \frac{T_4}{T_1} \left( \frac{P_1}{P_3} \right) \frac{k_h - 1}{k_h} = T_3 \frac{T_4}{T_1} \left( \frac{P_1}{P_3} \right) \frac{k_h - 1}{k_h} = \frac{T_3}{T_1} \left( \frac{P_1}{P_3} \right) \frac{k_h - 1}{k_h} = \frac{T_3}{T_1} \left( \frac{P_1}{P_3} \right) \frac{k_h - 1}{k_h} = \frac{T_3}{T_1} \left( \frac{P_1}{P_3} \right) \frac{k_h - 1}{k_h} \]  

(21)

Where \( k_c = C_{pc}/C_v \) and \( k_h = C_{ph}/C_v \) are specific heat ratios for cold and burned gases, respectively. Substituting Eq. (21) into Eq. (20) gives:

\[ \frac{w_{\text{net}}}{c_v T_1} = k_c - r - k_{cc} + \frac{c_{hv} k_h}{c_v} t \frac{(k_h - 1)(k_{cc} - 1)}{k_h} \left( 1 - \frac{1}{r} \right) \]  

(22)

The cycle efficiency can easily be derived from Eq. (22) in the same fashion as described before:

\[ \eta = \frac{\frac{w_{\text{net}}}{c_v T_1}}{c_v T_3 - c_v T_2} = \left[ k_c - r - k_{cc} + \frac{c_{hv} k_h}{c_v} t - \frac{c_{hv} k_h}{c_v} r^{-1} \right] \]  

(23)

Figure 10: Plot of ideal efficiency with constant and variable specific heats \( (k_c = 1.4, k_h = 1.33, R_c = R_h = R = 287 \frac{T}{kg_k}) \) versus pressure ratio for \( t = 4, 5, 6. \)

Equations (23) is plotted in Fig. 10 with pressure ratio as the independent variable for several values of temperature ratio, \( t = 4, 5, 6 \) and typical values of \( k_c = 1.4, \ k_h = 1.33 \) for air and burned gasses. A comparison is also made with the cold-air standard Atkinson cycle considered before (e.g. see Fig. 4). Results show that the variable specific heat model predicts lower efficiency which is expected with a more realistic model. Physically, for the same burner inlet and exit conditions, the variable specific model would demand more heat to be provide to the cycle, lowering the efficiency.

PERFORMANCE OF ATKINSON CYCLE WITH VARIABLE SPECIFIC HEATS INCLUDING COMPRESSION AND EXPANSION LOSSES

To make the model more comprehensive, an analysis with cold air and burned gas specific heats applied to a non-ideal cycle needs to be developed. First, an expression for net work output is developed as follows:

\[ w_{\text{net}} = c_{ph}(T_3 - T_4) - c_{pc}(T_2 - T_1) - (R_h T_3 - R_c T_2) \]
\[ = c_{ph} T_3 \left( 1 - \frac{T_4}{T_3} \right) - c_{pc} T_1 \left( \frac{T_2}{T_1} - 1 \right) - T_2 \left( R_h \frac{T_3}{T_2} - R_c \right) \]  

(24)

Applying isentropic efficiency expressions to Eq. (24) and normalizing by \( c_v T_1 \) yields:

\[ \frac{w_{\text{net}}}{c_v T_1} = c_{ph} T_3 \left[ \frac{\eta}{c_v} \left( 1 - \frac{P_4}{P_3} \frac{k_h - 1}{k_h} \right) - k_c \left( \frac{k_c - 1}{\eta_c} - 1 \right) \right] \]  

\[ - 1 \left[ \frac{R_h}{c_v} \left( \frac{k_c - 1}{\eta_c} - 1 \right) + 1 \right] \]  

(25)
Using part of the derivation for Eq. (21), \( P_3 / P_2 \) can be written in terms of \( r \) and \( t \). Hence, Eq. (25) can be written as:

\[
\frac{w_{net}}{c_{vc} T_1} = \frac{c_{ph}}{c_{vc}} \left[ \eta_t \left( 1 - \left( \frac{k_c - 1}{r - k_c} - 1 \right) \frac{k_h - 1}{\eta_c} + 1 \right) \right] \quad (26)
\]

The cycle efficiency follows simply by multiplying Eq. (26) by \( c_{vc} T_1 \) and dividing by the definition of \( q_{in} \) adjusted to pre and post combustion specific heats:

\[
\eta = \frac{\frac{w_{net}}{c_{vc} T_1}}{\frac{c_{ph}}{c_{vc}} t} \quad (27)
\]

Figure 11 shows a plot of Eq. (27) compared to results obtained in Fig. 8 for a cold air cycle. From the graph, it appears that the model with variable specific heats predicts less efficiency at lower pressure ratios. However if the pressure ratio becomes high enough, the efficiency of the multiple specific heat model becomes higher than the cold air standard model, as the decay of the curve is much shallower than the cold air standard model.

Lastly, Fig. 12 compares all four models of efficiencies (cold air ideal, ideal with variable specific heat effects, cold air non-ideal, and non-ideal with variable specific heat effects) for \( t = 4 \). This allows for prospective on how the efficiency of each model compares with the others.

As expected, the cold-air ideal model is the most efficient, with the maximum efficiency possible for the cycle being the Carnot efficiency. The ideal model with variable specific heats is more realistic and is the next most efficient. The interesting part of this plot, however, is how the models with losses intersect each other, and the air standard with losses model becomes less efficient at higher pressure ratios.

**SUMMARY**

In this study, the thermodynamics of the Atkinson cycle was analyzed using pure thermodynamics relationships. The analysis explored multiple models using graphical representations to interpret the results. The findings mapped out the optimum net work output and cycle efficiency for showing the relationship between these two quantities, and how the pressure ratio and operating temperature affects them. The results show that the effects of variable specific-heats of working fluid on the cycle performance are significant, and should be considered in practical cycle analysis. The results obtained in this paper may provide guidelines for the design of practical engines operating on the Atkinson cycle.
**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$C_p$</td>
<td>Specific heat at constant pressure</td>
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<tr>
<td>$C_v$</td>
<td>Specific heat at constant volume</td>
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<tr>
<td>$k$</td>
<td>Specific heat ratio</td>
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<td>$R$</td>
<td>Ideal gas constant</td>
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<tr>
<td>$C_{pc}$</td>
<td>Specific heat at constant pressure for cold air</td>
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<tr>
<td>$C_{ph}$</td>
<td>Specific heat at constant pressure for burned gas</td>
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<tr>
<td>$t$</td>
<td>Temperature ratio</td>
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<td>$r$</td>
<td>Compression pressure ratio</td>
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<td>$\eta_c$</td>
<td>Compression isentropic efficiency</td>
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<td>$(r_{max})_w$</td>
<td>Largest pressure ratio corresponding to where net work output becomes zero</td>
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<tr>
<td>$(r_{opt})_\eta$</td>
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<tr>
<td>$(r_{max})_\eta$</td>
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**REFERENCES**


