Thermal Buckling of Joined Conical-Conical FGM Shells

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Abstract—Thermal buckling of a joined conical-conical shell system is investigated in this research. It is assumed that the shell is made of functionally graded materials (FGMs) whose thermomechanical properties vary continuously through the thickness direction. Thickness of both shells are equal. First order theory of shells is accompanied with the Donnell type of kinematic assumptions to establish the general equilibrium equations and associated boundary and continuity conditions with the aid of virtual displacement principle. The resulting system of equations are discreted using the semi-analytical generalized differential quadrature method (GDQM). Considering clamped and simply supported types of boundary conditions for the shell ends and intersection continuity conditions, an eigenvalue problem is established to examine the critical temperature as well as the associated mode shapes. After proving the efficiency and validity of the present method for the case of single FGM conical shell, some parametric studies are carried out for joined shells made of the FGMs.

I. INTRODUCTION

Stability investigation of thin conical shells have vast applications in different fields of engineering. With the introduction of functionally graded materials (FGMs), recent researches on thermal stability of conical shells are focused on those made of FGMs. Bhangale et al. [1] applied a semi-analytical finite element method to the thermal buckling of conical shells using the first-order shell theory. The shell is divided into many sub-layers through the thickness direction where each of them is assumed to be isotropic and homogeneous. Prebuckling deformations of the shell are obtained employing the linear bending deformation assumptions. Akbari et al. [2] used the generalized differential quadrature method to investigate the thermal buckling of FGM conical shell with arbitrary edge supports. Classical shell theory is used and Prebuckling deformations of the shell are obtained using the linear membrane approach. There is no result on the thermal buckling of joined conical shells in the open literature. The present study deals with such subject for joined shells made of FGMs using the first order shell theory.

II. GOVERNING EQUATIONS

Consider a joined circular conical-conical shell made of FGMs of uniform thickness h, end radii R_1, R_3, intersection radius R_2, slanted lengths L_1 and L_2, and vertex half angles α_1 and α_2. Meridional, circumferential, and normal directions
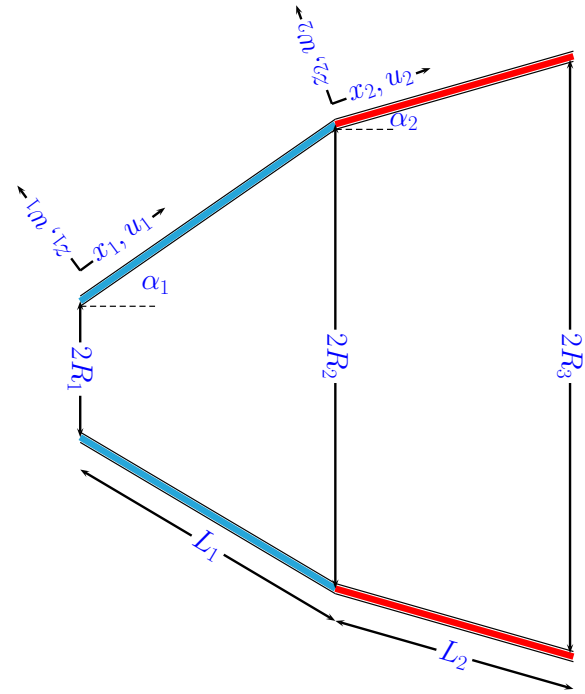

Fig. 1. Geometric parameters and coordinate system sign of a joined conical-conical closed shell.

of each conical shell are denoted by $0 \leq x^i \leq L_i$, $i = 1, 2$, $0 \leq \theta \leq 2\pi$ and $-h/2 \leq z \leq +h/2$, respectively. The adopted coordinates system (x^i, θ, z), geometric characteristics, and sign convention of the joined shell are depicted in Fig. (1). Mechanical properties of the FGM shell should be obtained according to a homogenization technique, e.g. Voigt rule of mixture. In this study we assume that segments consist of the same constituents and the properties dispersion is the same for both segments. Each property of the shell may be expressed as

$$P(z, T) = P_c(T)V_c(z) + P_m(T)V_m(z) \qquad (1)$$

where P describes any properties of the shell and the subscripts m and c represent the properties of metal and ceramic constituents, respectively, and V indicates the volume fraction. Following Akbari et al. [3], a power law function may be used to represent the volume fractions of ceramic and metal through

the thickness such that

$$V_c = \left(\frac{1}{2} + \frac{z}{h}\right)^k, V_m = 1 - V_c \qquad (2)$$

where in the above equation, k is the power law index and dictates the distribution of material properties through the thickness. Material properties, as seen in Eq. (1), are assumed to be temperature dependent. Temperature dependency of the FGM constituents are frequently expressed based on the Touloukian formula [2] in which higher order dependency to the temperature is also included. Accordingly, each property of the metal or ceramic may be written in the form

$$P(T) = P_0(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \qquad (3)$$

Based on the first order shear deformation theory (FSDT) of shells, components of the displacement on a generic point may be represented according to the mid-surface characteristics such that

$$\overline{u}^i\left(x^i, \theta, z\right) = u^i\left(x^i, \theta\right) + z\varphi_x^i\left(x^i, \theta\right)$$
$$\overline{v}^i\left(x^i, \theta, z\right) = v^i\left(x^i, \theta\right) + z\varphi_\theta^i\left(x^i, \theta\right)$$
$$\overline{w}^i\left(x^i, \theta, z\right) = w^i\left(x^i, \theta\right) \qquad (4)$$

In the above equation $\overline{u}$, $\overline{v}$, and $\overline{w}$ are the meridional, circumferential, and through-the-thickness displacements, respectively. Here, u, v, and w are the meridional, circumferential, and through-the-thickness displacements of the mid-surface, respectively. Besides, φ_θ and φ_x are, respectively, the transverse normal rotations about the x and θ axes. Furthermore, herein and in all the rest, superscript i takes the values of 1 and 2 and is associated with the ith. shell segment. According to FSDT, the components of strain field on an arbitrary point of the conical shell may be obtained in terms of those belong to the mid-surface of the shell and change of curvatures. Consequently, one may write [3]

$$\left\{\begin{array}{c} \overline{\varepsilon}_{xx}^i \\ \overline{\varepsilon}_{\theta\theta}^i \\ \overline{\gamma}_{x\theta}^i \\ \overline{\gamma}_{xz}^i \\ \overline{\gamma}_{\theta z}^i \end{array}\right\} = \left\{\begin{array}{c} \varepsilon_{xx}^i \\ \varepsilon_{\theta\theta}^i \\ \gamma_{x\theta}^i \\ \gamma_{xz}^i \\ \gamma_{\theta z}^i \end{array}\right\} + z \left\{\begin{array}{c} \kappa_{xx}^i \\ \kappa_{\theta\theta}^i \\ \kappa_{x\theta}^i \\ \kappa_{xz}^i \\ \kappa_{\theta z}^i \end{array}\right\} \qquad (5)$$

where the components of the strain associated with the mid-surface of the shell are [4]

$$\varepsilon_{xx}^i = u_{,x^i}^i + \frac{1}{2}\left(w_{,x^i}^i\right)^2$$
$$\varepsilon_{\theta\theta}^i = \frac{v_{,\theta}^i}{r(x^i)} + \frac{\cos(\alpha_i)}{r(x^i)}w^i + \frac{\sin(\alpha_i)}{r(x^i)}u^i + \frac{1}{2r^2(x^i)}\left(w_{,\theta}^i\right)^2$$
$$\gamma_{x\theta}^i = \frac{u_{,\theta}^i}{r(x^i)} + v_{,x^i}^i - \frac{\sin(\alpha_i)}{r(x^i)}v^i + \frac{1}{r(x^i)}w_{,x^i}^i w_{,\theta}^i$$
$$\gamma_{xz}^i = w_{,x^i}^i + \varphi_x^i$$
$$\gamma_{\theta z}^i = \frac{w_{,\theta}^i}{r(x^i)} - \frac{\cos(\alpha_i)}{r(x^i)}v^i + \varphi_\theta^i \qquad (6)$$

and the components of change in curvature in the Donnell sense compatible with the FSDT are [4]

$$\kappa_{xx}^i = \varphi_{x,x^i}^i$$
$$\kappa_{\theta\theta}^i = \frac{\varphi_{\theta,\theta}^i}{r(x^i)} + \frac{\sin(\alpha_i)}{r(x^i)}\varphi_x^i$$
$$\kappa_{x\theta}^i = \frac{\varphi_{x,\theta}^i}{r(x^i)} + \varphi_{\theta,x^i}^i - \frac{\sin(\alpha_i)}{r(x^i)}\varphi_\theta^i$$
$$\kappa_{xz}^i = 0$$
$$\kappa_{\theta z}^i = 0 \qquad (7)$$

where in the above equations $()_{,x^i}$ and $()_{,\theta}$ denote the derivatives with respect to the meridian and circumferential directions of the shell, respectively. Furthermore, $r\left(x^i\right) = R_i + x^i \sin\left(\alpha_i\right)$ stands for the radius of the joined shell at each point along the length. For the case when material properties of the shell are linearly elastic, components of stress in terms of strains are evaluated as

$$\sigma_{xx}^i = Q_{11}\overline{\varepsilon}_{xx}^i + Q_{12}\overline{\varepsilon}_{\theta\theta}^i - (T - T_0)\alpha$$
$$\sigma_{\theta\theta}^i = Q_{12}\overline{\varepsilon}_{xx}^i + Q_{22}\overline{\varepsilon}_{\theta\theta}^i - (T - T_0)\alpha$$
$$\tau_{\theta z}^i = Q_{44}\overline{\gamma}_{\theta z}^i$$
$$\tau_{xz}^i = Q_{55}\overline{\gamma}_{xz}^i$$
$$\tau_{x\theta}^i = Q_{66}\overline{\gamma}_{x\theta}^i \qquad (8)$$

where Q_{ij}'s $(i, j = 1, 2, 4, 5, 6)$ are the reduced material stiffness coefficients and are obtained as follow

$$Q_{11} = Q_{22} = \frac{E(z)}{1 - \nu^2(z)}, \quad Q_{12} = \frac{\nu(z)E(z)}{1 - \nu^2(z)}$$
$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1 + \nu(z))} \qquad (9)$$

The components of stress resultants are obtained using the components of stress field as

$$\left\{\begin{array}{c} N_{xx}^i \\ N_{\theta\theta}^i \\ N_{x\theta}^i \end{array}\right\} = \int_{-h/2}^{+h/2} \left\{\begin{array}{c} \sigma_{xx}^i \\ \sigma_{\theta\theta}^i \\ \tau_{x\theta}^i \end{array}\right\} dz,$$

$$\left\{\begin{array}{c} M_{xx}^i \\ M_{\theta\theta}^i \\ M_{x\theta}^i \end{array}\right\} = \int_{-h/2}^{+h/2} z \left\{\begin{array}{c} \sigma_{xx}^i \\ \sigma_{\theta\theta}^i \\ \tau_{x\theta}^i \end{array}\right\} dz,$$

$$\left\{\begin{array}{c} Q_{xz}^i \\ Q_{\theta z}^i \end{array}\right\} = \int_{-h/2}^{+h/2} \left\{\begin{array}{c} \sigma_{xz}^i \\ \sigma_{\theta z}^i \end{array}\right\} dz \qquad (10)$$

Substitution of Eq. (8) into Eq. (10) with the simultaneous aid of Eqs. (5), (6), and (7) generates the stress resultants in terms of the mid-surface characteristics of the shell as

$$N_{xx}^i = A_{11}\varepsilon_{xx}^i + A_{12}\varepsilon_{\theta\theta}^i + B_{11}\kappa_{xx}^i + B_{12}\kappa_{\theta\theta}^i - N^T$$
$$N_{\theta\theta}^i = A_{12}\varepsilon_{xx}^i + A_{22}\varepsilon_{\theta\theta}^i + B_{12}\kappa_{xx}^i + B_{22}\kappa_{\theta\theta}^i - N^T$$
$$N_{x\theta}^i = A_{66}\gamma_{x\theta}^i + B_{66}\kappa_{x\theta}^i$$
$$M_{xx}^i = B_{11}\varepsilon_{xx}^i + B_{12}\varepsilon_{\theta\theta}^i + D_{11}\kappa_{xx}^i + D_{12}\kappa_{\theta\theta}^i - M^T$$
$$M_{\theta\theta}^i = B_{12}\varepsilon_{xx}^i + B_{22}\varepsilon_{\theta\theta}^i + D_{12}\kappa_{xx}^i + D_{22}\kappa_{\theta\theta}^i - M^T$$
$$M_{x\theta}^i = B_{66}\gamma_{x\theta}^i + D_{66}\kappa_{x\theta}^i$$
$$Q_{\theta z}^i = A_{44}\gamma_{\theta z}^i$$
$$Q_{xz}^i = A_{55}\gamma_{xz}^i \qquad (11)$$

In the above equation, the constant coefficients A_{ij}, B_{ij}, and D_{ij} indicate the stretching, bending-stretching, and bending stiffnesses, respectively, which are calculated by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-0.5h}^{+0.5h} (Q_{ij}, zQ_{ij}, z^2 Q_{ij})dz \qquad (12)$$

Besides, N^T and M^T are the thermal force and thermal moment resultants, which are given by

$$N^T = \int_{-0.5h}^{+0.5h} \frac{1}{1-\nu(z,T)} E(z,T)\alpha(z,T)(T-T_0)dz$$

$$M^T = \int_{-0.5h}^{+0.5h} \frac{z}{1-\nu(z,T)} E(z,T)\alpha(z,T)(T-T_0)dz \quad (13)$$

The complete set of nonlinear equilibrium equations and the associated boundary conditions may be obtained with the aid of static version of virtual displacements. For a conical shell that is subjected to pure thermal loading, the total energy of the shell in an equilibrium position takes the form

$$\delta U^i = \int_0^{L_i} \int_0^{2\pi} \int_{-h/2}^{+h/2} (\sigma_{xx}^i \delta\varepsilon_{xx}^i + \sigma_{\theta\theta}^i \delta\varepsilon_{\theta\theta}^i$$

$$+\tau_{x\theta}^i \delta\gamma_{x\theta}^i + \tau_{xz}^i \delta\gamma_{xz}^i + \tau_{\theta z}^i \delta\gamma_{\theta z}^i) r(x^i)dzd\theta dx^i \qquad (14)$$

Integrating the above expression with respect to z and performing the Green-Guass theorem to relieve the virtual displacement gradients results in the expressions for the nonlinear equilibrium equations of the FGM conical shell as

$$N_{xx,x^i}^i + \frac{N_{x\theta,\theta}^i}{r(x^i)} + \frac{\sin(\alpha_i)}{r(x^i)}(N_{xx}^i - N_{\theta\theta}^i) = 0$$

$$\frac{N_{\theta\theta,\theta}^i}{r(x^i)} + N_{x\theta,x^i}^i + 2\frac{\sin(\alpha_i)}{r(x^i)}N_{x\theta}^i + \frac{\cos(\alpha_i)}{r(x^i)}Q_{\theta z}^i = 0$$

$$Q_{xz,x^i}^i + \frac{1}{r(x^i)}Q_{\theta z,\theta}^i + \frac{\sin(\alpha_i)}{r(x^i)}Q_{xz}^i - \frac{\cos(\alpha_i)}{r(x^i)}N_{\theta\theta}^i$$

$$-\frac{1}{r(x^i)}\left(r(x^i)N_{xx}^i w_{,x^i}^i + N_{x\theta}^i w_{,\theta}^i\right)_{,x^i}$$

$$-\frac{1}{r(x^i)}\left(\frac{1}{r(x^i)}N_{\theta\theta}^i w_{,\theta}^i + N_{x\theta}^i w_{,x^i}^i\right)_{,\theta} = 0$$

$$M_{xx,x^i}^i + \frac{1}{r(x^i)}M_{x\theta,\theta}^i + \frac{\sin(\alpha_i)}{r(x^i)}\left(M_{xx}^i - M_{\theta\theta}^i\right) - Q_{xz}^i = 0$$

$$M_{x\theta,x^i}^i + \frac{1}{r(x^i)}M_{\theta\theta,\theta}^i + \frac{2\sin(\alpha_i)}{r(x^i)}M_{x\theta}^i - Q_{\theta z}^i = 0 \qquad (15)$$

The complete set of boundary conditions for each side of the shell may be written as

$$N_{xx}^i \delta u^i = 0$$
$$N_{x\theta}^i \delta v^i = 0$$
$$\left(Q_{xz}^i + N_{xx}^i w_{,x^i}^i + \frac{1}{r(x^i)}N_{x\theta}^i w_{,\theta}^i\right)\delta w^i = 0$$
$$M_{xx}^i \delta\varphi_x^i = 0$$
$$M_{x\theta}^i \delta\varphi_\theta^i = 0 \qquad (16)$$

III. PREBUCKLING SOLUTION, LINEAR MEMBRANE APPROACH

Linear membrane approach is used to find the prebuckling forces. In this approach, the von-Karman improvements as

well as bending moments and curvatures are excluded from Eqs. (15) and (16). Therefore, the pre-buckling deformations of shell are obtained via the solution of the following equations [2]

$$N_{xx,x^i}^i + \frac{\sin(\alpha_i)}{r(x^i)}(N_{xx}^i - N_{\theta\theta}^i) = 0$$

$$\frac{\cos(\alpha_i)}{r(x^i)}N_{\theta\theta}^i = 0 \qquad (17)$$

along with the next boundary conditions [2]

$$N_{xx}^i \delta u^i = 0 \qquad (18)$$

in which

$$N_{xx}^i = A_{11}u_{,x^i}^i + A_{12}\left(\frac{\cos(\alpha_i)}{r(x^i)}w^i + \frac{\sin(\alpha_i)}{r(x^i)}u^i\right) - N^T$$

$$N_{\theta\theta}^i = A_{12}u_{,x^i}^i + A_{22}\left(\frac{\cos(\alpha_i)}{r(x^i)}w^i + \frac{\sin(\alpha_i)}{r(x^i)}u^i\right) - N^T \qquad (19)$$

The linear membrane pre-buckling solution may be used effectively for moderately long shells, where the effect of edge zone function near the edges of the shell is not dominant. In this research, this simple approach is employed to obtain the pre-buckling deformations. Based on the second of equilibrium equations (17), the circumferential stress resultant in pre-buckling state is equal to zero

$$N_{\theta\theta 0}^i = 0 \qquad (20)$$

Here, a subscript '0' indicates the pre-buckling characteristics. With the aid of the above equation and referring to the basic strain-displacement relations (19), the lateral deflection of the shell in pre-buckling state takes the form

$$w_0^i = \frac{r(x^i)}{A_{22}\cos(\alpha_i)}\left(N^T - A_{12}u_{0,x^i}^i - \frac{A_{22}\sin(\alpha_i)}{r(x^i)}u_0^i\right) \qquad (21)$$

Solution of the first equilibrium equation (17) in conjunction with Eq. (20) yields

$$N_{xx0}^i = \frac{C_1}{r(x^i)} \qquad (22)$$

After expanding the above expression in terms of u_0^i and w_0^i and usage of Eq. (21), a first order differential equation in terms of u_0^i is obtained. Under the exact solution of such equation along with the immovability conditions $u_0^i(0) = u_0^i(L_2) = 0$ (associated with Eq. (18)) one obtains the axial compressive force of the shell as

$$N_{xx0}^i = \frac{(A_{12} - A_{22})\sin(\alpha_i)L_i}{A_{22}r(x^i)\ln(1 + L_i\sin(\alpha_i)/R_i)}N^T \qquad (23)$$

IV. STABILITY EQUATIONS

Linearized stability equations are obtained by the concept of adjacent equilibrium criterion. According to this criterion, which is based on the perturbation technique, the components of displacements on primary equilibrium path are perturbed infinitesimally to establish an adjacent equilibrium position.

Therefore, displacement components associated with the secondary equilibrium path are

$$
\left\{
\begin{array}{c}
u^i(x,\theta) \\
v^i(x,\theta) \\
w^i(x,\theta) \\
\varphi_x^i(x,\theta) \\
\varphi_\theta^i(x,\theta)
\end{array}
\right\}
=
\left\{
\begin{array}{c}
u_0^i(x) \\
v_0^i(x) \\
w_0^i(x) \\
\varphi_{x0}^i(x,\theta) \\
\varphi_{\theta0}^i(x,\theta)
\end{array}
\right\}
+
\left\{
\begin{array}{c}
u_1^i(x,\theta) \\
v_1^i(x,\theta) \\
w_1^i(x,\theta) \\
\varphi_{x1}^i(x,\theta) \\
\varphi_{\theta1}^i(x,\theta)
\end{array}
\right\}
\quad (24)
$$

where displacement components with subscript 1 are infinitesimal and nonzero displacements. After substitution of the above equation into Eq. (11), the incremental values of stress resultants are obtained. Since the incremental displacements are small enough, stability equations associated with the equilibrium equations (15) are as follow

$$
N_{xx1,x^i}^i + \frac{N_{x\theta1,\theta}^i}{r(x^i)} + \frac{\sin(\alpha_i)}{r(x^i)}(N_{xx1}^i - N_{\theta\theta1}^i) = 0
$$

$$
\frac{N_{\theta\theta1,\theta}^i}{r(x^i)} + N_{x\theta1,x^i}^i + 2\frac{\sin(\alpha_i)}{r(x^i)}N_{x\theta1}^i + \frac{\cos(\alpha_i)}{r(x^i)}Q_{\theta z1}^i = 0
$$

$$
Q_{xz1,x^i}^i + \frac{1}{r(x^i)}Q_{\theta z1,\theta}^i + \frac{\sin(\alpha_i)}{r(x^i)}Q_{xz1}^i - \frac{\cos(\alpha_i)}{r(x^i)}N_{\theta\theta1}^i
$$

$$
- \frac{1}{r(x^i)}\left(r(x^i)N_{xx0}^i w_{1,x^i}^i + N_{x\theta0}^i w_{1,\theta}^i\right)_{,x^i}
$$

$$
- \frac{1}{r(x^i)}\left(\frac{1}{r(x^i)}N_{\theta\theta0}^i w_{1,\theta}^i + N_{x\theta0}^i w_{1,x^i}^i\right)_{,\theta} = 0
$$

$$
M_{xx1,x^i}^i + \frac{1}{r(x^i)}M_{x\theta1,\theta}^i + \frac{\sin(\alpha_i)}{r(x^i)}\left(M_{xx1}^i - M_{\theta\theta1}^i\right)
$$

$$
- Q_{xz1}^i = 0
$$

$$
M_{x\theta1,x^i}^i + \frac{1}{r(x^i)}M_{\theta\theta1,\theta}^i + \frac{2\sin(\alpha_i)}{r(x^i)}M_{x\theta1}^i - Q_{\theta z1}^i = 0 \quad (25)
$$

Using (16) the complete set of incremental boundary conditions for each side of the shell take the form

$$
N_{xx1}^i \delta u_1^i = 0
$$

$$
N_{x\theta1}^i \delta v_1^i = 0
$$

$$
\left(Q_{xz1}^i + N_{xx0}^i w_{1,x^i}^i + \frac{1}{r(x^i)}N_{x\theta0}^i w_{1,\theta}^i\right)\delta w_1^i = 0
$$

$$
M_{xx1}^i \delta \varphi_{x1}^i = 0
$$

$$
M_{x\theta}^i \delta \varphi_{\theta1}^i = 0 \quad (26)
$$

For the two ends of the conical shell, various types of boundary conditions may be defined. In this study each of the edges $x_1 = 0$ and $x_2 = L_2$ may be clamped (C) or simply supported (S).

$$
C \; : \; u_1^i = v_1^i = w_1^i = \varphi_{x1}^i = \varphi_{\theta1}^i = 0
$$

$$
S \; : \; u_1^i = v_1^i = w_1^i = M_{xx1}^i = \varphi_{\theta1}^i = 0 \quad (27)
$$

At the intersection of the shell system, the continuity of displacement components as well as the force and moment resultants should be satisfied. The compatibility of the displacements at the intersection reads

$$
u_1^1 \cos(\alpha_1) - w_1^1 \sin(\alpha_1) = u_1^2 \cos(\alpha_2) - w_1^2 \sin(\alpha_2)
$$

$$
u_1^1 \sin(\alpha_1) + w_1^1 \cos(\alpha_1) = u_1^2 \sin(\alpha_2) + w_1^2 \cos(\alpha_2)
$$

$$
v_1^1 = v_1^2
$$

$$
\varphi_{x1}^1 = \varphi_{x1}^2
$$

$$
\varphi_{\theta1}^1 = \varphi_{\theta1}^2 \quad (28)
$$

and similarly the compatibility of the stress resultants at the intersection results is

$$
N_{xx1}^1 \cos(\alpha_1) - Q_{xz1}^1 \sin(\alpha_1) = N_{xx1}^2 \cos(\alpha_2) - Q_{xz1}^2 \sin(\alpha_2)
$$

$$
N_{xx1}^1 \sin(\alpha_1) + Q_{xz1}^1 \cos(\alpha_1) = N_{xx1}^2 \sin(\alpha_2) + Q_{xz1}^2 \cos(\alpha_2)
$$

$$
M_{xx1}^1 = M_{xx1}^2
$$

$$
N_{x\theta1}^1 = N_{x\theta1}^2
$$

$$
M_{x\theta1}^1 = M_{x\theta1}^2 \quad (29)
$$

V. SOLUTION PROCEDURE

The stability equations (25) and boundary conditions (27) may be expressed in terms of the incremental displacements $u_1, v_1, w_1, \varphi_{x1}$ and $\varphi_{\theta1}$. To this end, linearized expansion of stress resultant in terms of displacements from Eq. (11) should be settled into Eqs. (25) and (27). Referring to the definition of normal force and bending moment resultants from Eq. (11) and the motion equations (15), the following separation of variables exactly satisfies the periodicity conditions of the field variables and is also compatible with the motion equations (11) and matching conditions (28) and (29).

$$
u_1^i(x^i,\theta) = \sin(n\theta)U^i(x^i)
$$

$$
v_1^i(x^i,\theta) = \cos(n\theta)V^i(x^i)
$$

$$
w_1^i(x^i,\theta) = \sin(n\theta)W^i(x^i)
$$

$$
\varphi_{x1}^i(x^i,\theta) = \sin(n\theta)\Phi_x^i(x^i)
$$

$$
\varphi_{\theta1}^i(x^i,\theta) = \cos(n\theta)\Phi_\theta^i(x^i) \quad (30)
$$

where in the above equation n is the wave number through the circumferential direction. Substitution of the above equation into the stability equations (25) results into new ten coupled ordinary differential equations in terms of the unknown through-the-meridian functions $U^i(x^i), V^i(x^i), W^i(x^i), \Phi_x^i(x^i)$ and $\Phi_\theta^i(x^i)$. The transformed equations and the associated boundary conditions for the i-th. segment are given here (for the sake of simplicity, the superscript i is dropped out). The stability equation in axial direction is

$$
A_{11}U_{,xx} + \frac{A_{12}}{r(x)}\left(\sin(\alpha)U_{,x} - nV_{,x} + \cos(\alpha)W_{,x}\right) +
$$

$$
B_{11}\Phi_{x,xx} + \frac{B_{12}}{r(x)}\left(\sin(\alpha)\Phi_{x,x} - n\Phi_{\theta,x}\right) -
$$

$$
\frac{B_{12}\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right) +
$$

$$
\frac{A_{66}}{r^2(x)}\left(-n^2 U + n\sin(\alpha)V - nr(x)V_{,x}\right) +
$$

$$
\frac{B_{66}}{r^2(x)}\left(-n^2\Phi_x + n\sin(\alpha)\Phi_\theta - nr(x)\Phi_{\theta,x}\right) +
$$

$$
\frac{(A_{11} - A_{12})}{r(x)}\sin(\alpha)U_{,x} -
$$

$$
\frac{A_{22}\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)U - nV + \cos(\alpha)W\right) +
$$

$$
\frac{(B_{11} - B_{12})\sin(\alpha)}{r(x)}\Phi_{x,x} +
$$

$$
\frac{(B_{12} - B_{22})\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right) = 0 \quad (31)
$$

The stability equation in circumferential direction is

$$\frac{A_{12}}{r(x)}nU_{,x} + \frac{A_{22}}{r^2(x)}\left(nU\sin(\alpha) - n^2V + n\cos(\alpha)W\right) +$$

$$\frac{B_{12}}{r(x)}n\Phi_{x,x} + \frac{B_{22}}{r^2(x)}\left(n\sin(\alpha)\Phi_x - n^2\Phi_\theta\right) +$$

$$\frac{A_{66}\sin(\alpha)}{r^2(x)}\left(nU - \sin(\alpha)V + r(x)V_{,x}\right) +$$

$$\frac{B_{66}\sin(\alpha)}{r^2(x)}\left(n\Phi_x - \sin(\alpha)\Phi_\theta + r(x)\Phi_{\theta,x}\right) +$$

$$\frac{A_{66}}{r(x)}\left(nU_{,x} + r(x)V_{,xx}\right) + \frac{B_{66}}{r(x)}\left(n\Phi_{x,x} + r(x)\Phi_{\theta,xx}\right) +$$

$$\frac{A_{44}\cos(\alpha)}{r^2(x)}\left(-\cos(\alpha)V + r(x)\Phi_\theta + nW\right) = 0 \qquad (32)$$

The stability equation in transverse direction is

$$\frac{-A_{12}\cos(\alpha)}{r(x)}U_{,x} -$$

$$\frac{A_{22}\cos(\alpha)}{r^2(x)}\left(\sin(\alpha)U - nV + \cos(\alpha)W\right) -$$

$$\frac{B_{12}}{r(x)}\cos(\alpha)\Phi_{x,x} - \frac{B_{22}\cos(\alpha)}{r^2(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right) +$$

$$\frac{A_{55}\sin(\alpha)}{r(x)}\left(\Phi_x + W_{,x}\right) + \kappa A_{55}\left(\Phi_{x,x} + W_{,xx}\right) +$$

$$\frac{A_{44}}{r^2(x)}\left(n\cos(\alpha)V - nr(x)\Phi_\theta - n^2W\right) -$$

$$\frac{(A_{12} - A_{22})\sin(\alpha)LN^T}{A_{22}\ln(1 + L\sin(\alpha)/R_1)}W_{,xx} = 0 \qquad (33)$$

The stability equation of moment resultants about axial direction is

$$B_{11}U_{,xx} + \frac{B_{12}}{r(x)}\left(\sin(\alpha)U_{,x} - nV_{,x} + \cos(\alpha)W_{,x}\right) +$$

$$D_{11}\Phi_{,xx} + \frac{D_{12}}{r(x)}\left(\sin(\alpha)\Phi_{x,x} - n\Phi_{\theta,x}\right) -$$

$$\frac{D_{12}\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right) +$$

$$\frac{B_{66}}{r^2(x)}\left(-n^2U + n\sin(\alpha)V - nr(x)V_{,x}\right) +$$

$$\frac{D_{66}}{r^2(x)}\left(-n^2\Phi_x + n\sin(\alpha)\Phi_\theta - nr(x)\Phi_{\theta,x}\right) +$$

$$\frac{(B_{11} - B_{12})\sin(\alpha)}{r(x)}U_{,x} -$$

$$\frac{B_{22}\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)U - nV + \cos(\alpha)W\right) +$$

$$\frac{(D_{11} - D_{12})\sin(\alpha)}{r(x)}\Phi_{x,x} +$$

$$\frac{(D_{12} - D_{22})\sin(\alpha)}{r^2(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right) -$$

$$A_{55}\left(\Phi_x + W_{,x}\right) = 0 \qquad (34)$$

The stability equation of moment resultants about circumferential direction is

$$\frac{B_{66}}{r(x)}\left(nU_{,x} + r(x)V_{,xx}\right) + \frac{D_{66}}{r(x)}\left(n\Phi_{x,x} + r(x)\Phi_{\theta,xx}\right) +$$

$$\frac{B_{12}}{r(x)}nU_{,x} + \frac{B_{22}}{r^2(x)}\left(n\sin(\alpha)U - n^2V + n\cos(\alpha)W\right) +$$

$$\frac{D_{12}}{r(x)}n\Phi_{x,x} + \frac{D_{22}}{r^2(x)}\left(n\sin(\alpha)\Phi_x - n^2\Phi_\theta\right) +$$

$$\frac{B_{66}\sin(\alpha)}{r^{2(x)}}\left(nU - \sin(\alpha)V + r(x)V_{,x}\right) +$$

$$\frac{D_{66}\sin(\alpha)}{r^2(x)}\left(n\Phi_x - \sin(\alpha)\Phi_\theta + r(x)\Phi_{\theta,x}\right) -$$

$$\frac{A_{44}}{r(x)}\left(-V\cos(\alpha) + r(x)\Phi_\theta + nW\right) = 0 \qquad (35)$$

Similarly, one should interpret the boundary conditions (27) with the aid of variable change (30). While the transformation of essential boundary conditions is straightforward, the natural type of boundary conditions after change of variables (30) take the following form

$$N_{xx} = A_{11}U_{,x} + \frac{A_{12}}{r(x)}\left(\sin(\alpha)U - nV + \cos(\alpha)W\right) +$$

$$B_{11}\Phi_{x,x} + \frac{B_{12}}{r(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right)$$

$$N_{x\theta} = \frac{A_{66}}{r(x)}\left(nU - \sin(\alpha)V + r(x)V_{,x}\right) +$$

$$\frac{B_{66}}{r(x)}\left(n\Phi_x - \sin(\alpha)\Phi_\theta + r(x)\Phi_{\theta,x}\right)$$

$$M_{xx} = B_{11}U_{,x} + \frac{B_{12}}{r(x)}\left(\sin(\alpha)U - nV + \cos(\alpha)W\right) +$$

$$D_{11}\Phi_{x,x} + \frac{D_{12}}{r(x)}\left(\sin(\alpha)\Phi_x - n\Phi_\theta\right)$$

$$M_{x\theta} = \frac{B_{66}}{r(x)}\left(nU - \sin(\alpha)V + r(x)V_{,x}\right) +$$

$$\frac{D_{66}}{r(x)}\left(n\Phi_x - \sin(\alpha)\Phi_\theta + r(x)\Phi_{\theta,x}\right)$$

$$Q_{xz} = A_{55}\left(\Phi_x + W_{,x}\right) \qquad (36)$$

As expected, Eqs. (31) to (35) along with a proper choice of boundary and matching conditions results in a system of homogeneous equations. To solve the system of equations as an eigenvalue problem, the GDQ method is implemented to transform the ordinary differential equations (31)-(35) into a new linear algebraic equations. The GDQ method is quietly well-known and its details are not repeated herein. Meanwhile one may refer to [5] for more details.

VI. NUMERICAL RESULT AND DISCUSSION

1) Comparison studies: In this section, the critical buckling temperature $T_{cr}(^\circ C)$ of a class of conical shell made of $SUS304$ and $Al2O3$ is evaluated and compared with the results of Bhangale et al. [1] in Table I. Thermomechanical properties of both of the constituents are highly temperature dependent. Temperature-dependent coefficients for $SUS304$ and Al_2O_3 are given in reference [2].

TABLE I. $T_{cr}[^oC]$ OF TEMPERATURE DEPENDENT $Al_2O_3/SUS304$ $C-C$ FGM CONICAL SHELLS. PROPERTIES OF THE SHELL ARE $L/h = 304.7896$.

k	$\alpha = 15^o$, $R_1/h = 252.5573$	
	Bhangale et al. [1]	Present
0.0	151.11	151.2 (-0.07%)
0.5	183.25	181.29 (1.07%)
1.0	202.89	199.47 (1.69%)
5.0	254.20	250.06 (1.63%)
10.0	270.86	267.24 (1.34%)
15.0	278.18	274.79 (1.22%)
100	294.95	290.90 (1.37%)
1000	296.35	293.00 (1.13%)
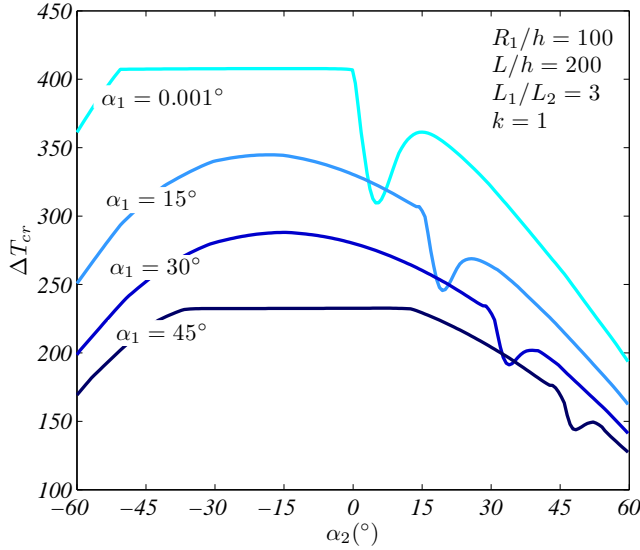
Fig. 2. Infuluence of semi vertex angle of second shell on critical buckling temperature difference of conical shells.
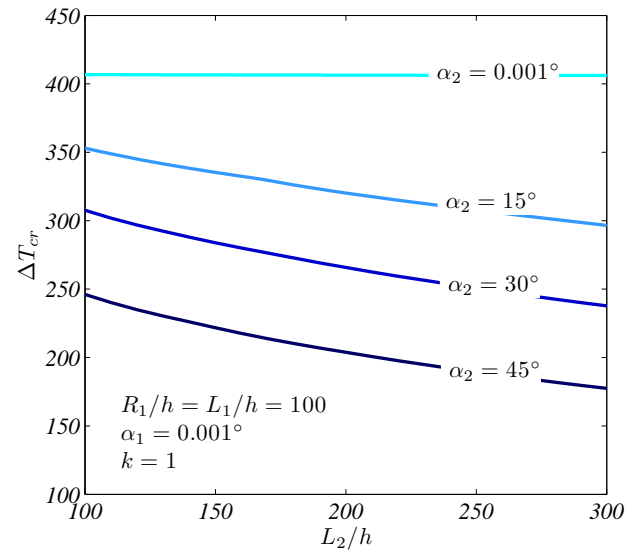

Fig. 3. Influence of length of second shell on critical buckling temperature difference of conical shells.
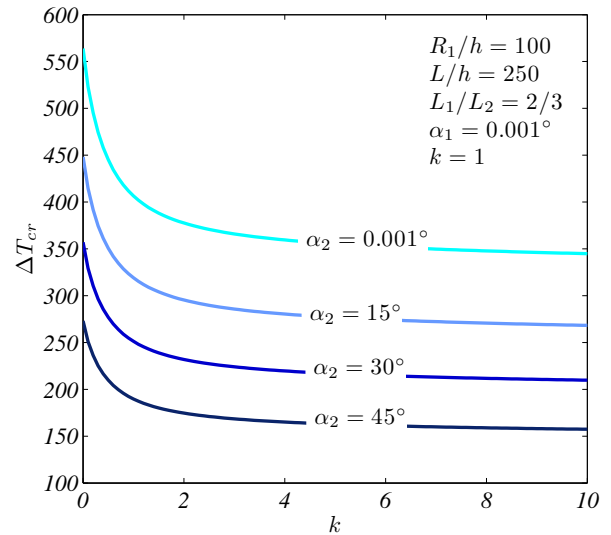

Fig. 4. Influence of length of second shell on critical buckling temperature difference of conical shells.

2) Parametric studies: In this section, a ceramic-metal FGM conical shell made of $SUS304$ and Si_3N_4 is considered. Thermomechanical properties of both of the constituents are highly temperature dependent. Temperature-dependent coefficients for $SUS304$ and Si_3N_4 are available in reference [2]. The critical buckling temperature difference of conical shells versus α_2 for different values of α_1 is shown in Fig. 2. It can be observed that where the two angles of the shell becomes equal, a sudden drop is occurred in the critical buckling temperature. The critical buckling temperature difference of conical shells versus L_2/h for different values of α_2 is illustrated in Fig. 3. It can be observed that critical buckling temperature difference decreases when L_2/h increases. Influence of power law index on critical buckling temperature difference of conical shell for different values of α_2 is illustrated in Fig. 4. It can be observed that critical buckling temperature difference decreases when power law index increases. Influence of edge support on the mode shape of conical shells are illustrated in Fig. 5.

VII. CONCLUSION

Linear thermal buckling of a joined conical shell made of functionally graded materials is investigated in this study. The shell is formulated using the Donnell kinematic assumptions accounting for the von-Karman type of geometrical nonlinearity. Material properties of the shell are obtained according to a power law expression and temperature dependency is regarded. Linear membrane Prebuckling approach is used. These equations are solved via a hybrid Fourier-GDQ method. It is shown that the critical buckling temperature of conical shells decreases permanently with the increase in shell length and power law index. Also, it is illustrated that when the semi vertex angle of two shells are approaching to a same value, the critical temperature difference decreases abruptly.

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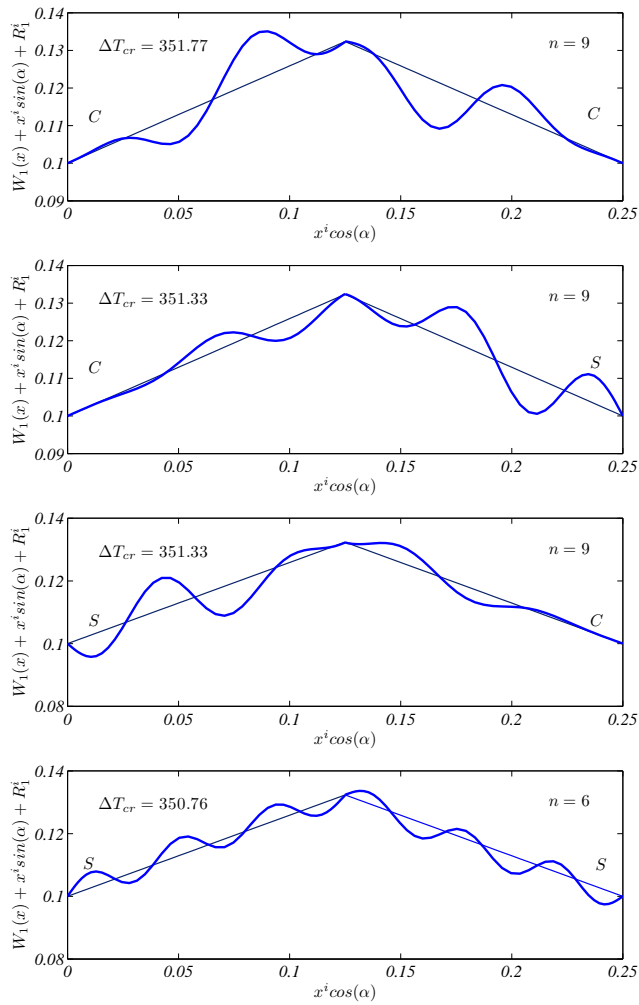
Fig. 5. Influence of edge support on the modeshape of conical shells. Properties of the shell are $k = 1, L/R_1 = 2.5, L_1/L_2 = 1, \alpha_1 = 15°, \alpha_2 = -15°$. ($n$ is the circumferential mode number).

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