

# Generalized Thermoelasticity of Rotating Disk

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**Abstract**—Generalized thermoelasticity of a disk which is rotating with constant angular velocity is analysed in this research. To account for the finite speed of thermal wave propagation, theory of one relaxation time known as the Lord-Shulman theory is considered. Two coupled equations, namely, the radial motion equation and the energy equation of the disk are obtained. Resulting equations are discreted by means of the generalised differential quadrature method along the disk radius. The resulting coupled equations are traced in time domain by means of the Newmark time marching scheme. Numerical results are provided to examine the propagation of thermal and mechanical waves in both stationary and rotating disks.

## I. INTRODUCTION

Due to the infinite speed of thermal wave propagation in conventional Fourier heat transfer, some nonclassical theories are developed. These theories which are known as the generalized thermoelasticity, or thermoelasticity with second sound effect, take into account the finite speed of thermal wave propagation. In the most simple theory, Lord and Shulman modified the conventional Fourier law by introducing a relaxation time and inserting the heat flux rate into the Fourier law [1]. This theory results in finite speed of thermal wave propagation.

Many researches are available on the response of disks and cylinders subjected to thermal shock within the framework of coupled or generalized thermoelasticity theories. Bagri and Eslami [2] analysed the thermal and displacement waves propagation in a disk based on the Lord-Shulman theory. Equations of motion and energy are established and solved via the Laplace transformation in time domain and finite elements formulation in radial direction. Bagri and Eslami also obtained a unified generalized thermoelasticity formulation for the cylinders and spheres made of homogeneous [3] or heterogeneous [4] materials. Hosseini and Abolbashari [5] presented an analytical solution to study the thermal and mechanical waves in a cylinder using the theory of coupled thermoelasticity without energy dissipation based on the Green-Naghdi model. The governing equations are transferred to the Laplace domain and solved via a series solution in Laplace domain.

This study is the modification of the previous research of the second author [2] where the body force induced due to rotation is taken into account. The resulting equations compatible with the single relaxation time theory of generalized thermoelasticity, namely radial wave propagation and energy equations are obtained in one-dimensional polar coordinates under plane stress conditions. A hollow disk made of an isotropic and homogeneous body under the action of thermal shock is considered. The equations are written in terms of the

nondimensional temperature and radial displacement. These equations are discreted by means of the generalized differential quadrature along the disk radius and traced in time by means of the Newmark time marching scheme. Numerical results are provided to explore the propagation and reflection of thermal and mechanical waves in a finite hollow disk.

## II. EQUATION OF MOTION

For the case of axisymmetric displacements in polar coordinates, with consideration of the body force induced due to rotation, the radial equation of motion takes the form [1]

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - \rho r \omega^2 = \rho \ddot{u} \quad (1)$$

In the above equation  $u$  is the radial displacement,  $\omega$  is the constant angular speed of the disk, and  $\sigma_{rr}$  and  $\sigma_{\theta\theta}$  are the radial and tangential stress components. For an isotropic homogeneous elastic body the components of stress are written in terms of strains as [2]

$$\begin{aligned} \sigma_{rr} &= 2\mu \varepsilon_{rr} + \bar{\lambda}(\varepsilon_{rr} + \varepsilon_{\theta\theta}) - \bar{\beta}(T - T_0) \\ \sigma_{\theta\theta} &= 2\mu \varepsilon_{\theta\theta} + \bar{\lambda}(\varepsilon_{rr} + \varepsilon_{\theta\theta}) - \bar{\beta}(T - T_0) \end{aligned} \quad (2)$$

Here,  $T_0$  is the reference temperature and  $T$  is the temperature profile. Moreover,  $\bar{\lambda}$  and  $\bar{\beta}$  are consistent with the plane stress conditions and are obtained as

$$\bar{\lambda} = \frac{2\mu}{\lambda + 2\mu} \lambda, \quad \bar{\beta} = \frac{2\mu}{\lambda + 2\mu} \beta \quad (3)$$

In the above equation,  $\mu$  and  $\lambda$  are the Lamé constants and  $\beta$  is thermoelastic parameter defined as  $\beta = (3\lambda + 2\mu)\alpha$ .

The components of strain in polar coordinates are written in terms of the radial displacement  $u$  as

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} \quad (4)$$

The radial equation of motion in terms of radial displacement is obtained when Eqs. (2) and (4) are inserted into Eq. (1). This equation is obtained as

$$(2\mu + \bar{\lambda}) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \rho r \omega^2 - \bar{\beta} \frac{\partial T}{\partial r} = \rho \ddot{u} \quad (5)$$

## III. ENERGY EQUATION

According to the Lord-Shulman theory, the conventional Fourier diffusion is modified by introduction of a relaxation time  $t_0$  as follows [1]

$$q + t_0 \dot{q} = -K \frac{\partial T}{\partial r} \quad (6)$$

where  $q$  is the heat flux and  $K$  is thermal conductivity. The heat balance for an element of a body relating the radial heat flux  $q$  to the rate of specific heat influx  $Q$  is [6]

$$\dot{Q} = -\frac{1}{r} \frac{\partial(rq)}{\partial r} \quad (7)$$

and from the second law of thermodynamics

$$\delta Q = T dS \quad (8)$$

The above equation, when is written in rate form, takes the form

$$\dot{Q} = T\dot{S} = T \left( \frac{\partial S}{\partial \varepsilon_{rr}} \dot{\varepsilon}_{rr} + \frac{\partial S}{\partial \varepsilon_{\theta\theta}} \dot{\varepsilon}_{\theta\theta} + \frac{\partial S}{\partial T} \dot{T} \right) \quad (9)$$

which simplifies to

$$\dot{Q} = T\bar{\beta}\dot{\varepsilon}_{rr} + T\bar{\beta}\dot{\varepsilon}_{\theta\theta} + c_\varepsilon\rho\dot{T} \quad (10)$$

Equations (6), (7), and (10) are combined together to obtain the most general form of the first law of thermodynamics in a rotating disk based on the Lord-Shulman theory

$$K \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - (1 + t_0 \frac{\partial}{\partial t}) (\bar{\beta}\dot{\varepsilon}_{rr}T + \bar{\beta}\dot{\varepsilon}_{\theta\theta}T) - (1 + t_0 \frac{\partial}{\partial t}) c_\varepsilon \rho \dot{T} = 0 \quad (11)$$

Substitution of strain field from Eq. (4) into the above equation results in

$$K \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - \bar{\beta}T \left( \frac{\partial \dot{u}}{\partial r} + \frac{\dot{u}}{r} \right) - t_0 \bar{\beta}T \left( \frac{\partial \dot{u}}{\partial r} + \frac{\dot{u}}{r} \right) - t_0 \bar{\beta}T \left( \frac{\partial \ddot{u}}{\partial r} + \frac{\ddot{u}}{r} \right) - c_\varepsilon \rho \dot{T} - t_0 c_\varepsilon \rho \ddot{T} = 0 \quad (12)$$

#### IV. EQUATIONS IN NON-DIMENSIONAL FORM

Equations (5) and (12) are transformed into the dimensionless form with the introduction of the following non-dimensional variables [7]

$$\begin{aligned} \hat{r} &= \frac{r}{l}, & \hat{t} &= \frac{tc_e}{l}, & \hat{t}_0 &= \frac{t_0 c_e}{l} \\ \hat{\theta} &= \frac{T - T_0}{T_0}, & \hat{\sigma}_{rr} &= \frac{\sigma_{rr}}{\beta T_0}, & \hat{\sigma}_{\theta\theta} &= \frac{\sigma_{\theta\theta}}{\beta T_0} \\ \hat{u} &= \frac{(\bar{\lambda} + 2\mu)u}{l\beta T_0}, & \hat{q} &= \frac{ql}{KT_0}, & \Omega &= \frac{\rho l^2 \omega^2}{\beta T_0} \end{aligned} \quad (13)$$

where in the above equation a symbol  $\hat{\quad}$  indicates the non-dimensional parameter. The parameter  $l$  indicates the characteristic length defined as  $l = K(\rho c_\varepsilon c_e)^{-1}$  and  $c_e$  is the speed of elastic wave propagation defined as  $c_e = \sqrt{(\bar{\lambda} + 2\mu)\rho^{-1}}$ .

With the aid of dimensionless variables (13), Eqs. (5), and (12) are represented in dimensionless form as

$$\begin{aligned} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \frac{\partial \theta}{\partial r} - \frac{\partial^2 u}{\partial t^2} - \Omega r &= 0 \\ \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) - Ct_0 \frac{\partial \theta}{\partial t} \left( \frac{\partial^2 u}{\partial r \partial t} + \frac{\partial u}{r \partial t} \right) - \frac{\partial \theta}{\partial t} - t_0 \frac{\partial^2 \theta}{\partial t^2} \\ - (1 + \theta)C \left( \frac{\partial^2 u}{\partial r \partial t} + \frac{\partial u}{r \partial t} + t_0 \frac{\partial^3 u}{\partial r \partial t^2} + t_0 \frac{\partial^2 u}{r \partial t^2} \right) &= 0 \end{aligned} \quad (14)$$

The second equation is the most general form of the energy equation based on the Lord-Shulman theory. This equation as seen is nonlinear in terms of the dimensionless temperature and radial displacement. For the case when temperature change is small enough in comparison with the reference temperature ( or according to dimensions presentation of temperature profile,  $\theta \ll 1$ ),  $1 + \theta$  is approximated by 1. Under such assumption, the linearised energy equation is achieved and Eq. (14) simplifies to

$$\begin{aligned} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \frac{\partial \theta}{\partial r} - \frac{\partial^2 u}{\partial t^2} - \Omega r &= 0 \\ \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) - \frac{\partial \theta}{\partial t} - t_0 \frac{\partial^2 \theta}{\partial t^2} \\ - C \left( \frac{\partial^2 u}{\partial r \partial t} + \frac{\partial u}{r \partial t} + t_0 \frac{\partial^3 u}{\partial r \partial t^2} + t_0 \frac{\partial^2 u}{r \partial t^2} \right) &= 0 \end{aligned} \quad (15)$$

where in the above equations, the symbol  $\hat{\quad}$  is dropped out of the equations for the sake of simplicity. Furthermore, the coefficient  $C$  is produced in the process of transferring the equation into dimensionless form and indicates the coupling effect. This coefficient is equal to

$$C = \frac{T_0 \bar{\beta}^2}{\rho c_\varepsilon (\bar{\lambda} + 2\mu)} \quad (16)$$

The non-dimensional components of the radial and circumferential stresses also in dimensionless form take the form

$$\begin{aligned} \sigma_{rr} &= \frac{2\mu}{\bar{\lambda} + 2\mu} \frac{\partial u}{\partial r} + \frac{\bar{\lambda}}{\bar{\lambda} + 2\mu} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) - \theta \\ \sigma_{\theta\theta} &= \frac{2\mu}{\bar{\lambda} + 2\mu} \frac{u}{r} + \frac{\bar{\lambda}}{\bar{\lambda} + 2\mu} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) - \theta \end{aligned} \quad (17)$$

Again note that the symbol  $\hat{\quad}$  is dropped out of the above relations for the sake of simplicity.

#### V. SOLUTION METHOD

The generalized differential quadrature method is used to discrete the motion and energy equations along the disk radius. Applying the generalized differential quadrature method to Eqs. (17) results in

$$\begin{aligned} \sum_{j=1}^N \left( C_{ij}^{(2)} + \frac{1}{r_i} C_{ij}^{(1)} - \frac{1}{r_i^2} C_{ij}^{(0)} \right) u_j - \sum_{j=1}^N C_{ij}^{(1)} \theta_j - \\ \sum_{j=1}^N C_{ij}^{(0)} \ddot{u}_j = \Omega r_i \\ \sum_{j=1}^N \left( C_{ij}^{(2)} + \frac{1}{r_i} C_{ij}^{(1)} \right) \theta_j - \sum_{j=1}^N C_{ij}^{(0)} \dot{\theta}_j - t_0 \sum_{j=1}^N C_{ij}^{(0)} \ddot{\theta}_j \\ - C \sum_{j=1}^N \left( C_{ij}^{(1)} + \frac{1}{r_i} C_{ij}^{(0)} \right) \dot{u}_j \\ - Ct_0 \sum_{j=1}^N \left( C_{ij}^{(1)} + \frac{1}{r_i} C_{ij}^{(0)} \right) \ddot{u}_j = 0 \end{aligned} \quad (18)$$

where in the above equations  $C_{ij}^{(r)}$ 's are the weighting coefficients associated with the  $r$ -th derivative and  $N$  is the number

of grid points. Furthermore,  $r_i$  is the position of  $i$ -th node in the generalized differential quadrature method. Distribution of the points is based on the well-known Chebyshev-Gauss-Lobatto method. For the case when the inner and outer radii of the disk in nondimensional form are denoted by  $a$  and  $b$ , distribution of the points is obtained as

$$r_i = \frac{a+b}{2} + \frac{a-b}{2} \cos\left(\frac{i-1}{N-1}\right), \quad i = 1, 2, \dots, N \quad (19)$$

The system of equations (18) in a matrix form may be represented as

$$\begin{bmatrix} M^{uu} & M^{u\theta} \\ M^{\theta u} & M^{\theta\theta} \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} C^{uu} & C^{u\theta} \\ C^{\theta u} & C^{\theta\theta} \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} K^{uu} & K^{u\theta} \\ K^{\theta u} & K^{\theta\theta} \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = \begin{Bmatrix} F^u \\ F^\theta \end{Bmatrix} \quad (20)$$

where the elements of mass, damping, stiffness matrices, and force vector are as follow

$$\begin{aligned} M_{ij}^{uu} &= -C_{ij}^{(0)} \\ M_{ij}^{u\theta} &= 0 \\ C_{ij}^{uu} &= 0 \\ C_{ij}^{u\theta} &= 0 \\ K_{ij}^{uu} &= C_{ij}^{(2)} + \frac{1}{r_i} C_{ij}^{(1)} - \frac{1}{r_i^2} C_{ij}^{(0)} \\ K_{ij}^{u\theta} &= -C_{ij}^{(1)} \\ M_{ij}^{\theta u} &= -C_{ij}^{(1)} + \frac{1}{r_i} C_{ij}^{(0)} \\ M_{ij}^{\theta\theta} &= -t_0 C_{ij}^{(0)} \\ C_{ij}^{\theta u} &= -C_{ij}^{(1)} + \frac{1}{r_i} C_{ij}^{(0)} \\ C_{ij}^{\theta\theta} &= -C_{ij}^{(0)} \\ K_{ij}^{\theta u} &= 0 \\ K_{ij}^{\theta\theta} &= C_{ij}^{(2)} + \frac{1}{r_i} C_{ij}^{(1)} \\ F_i^u &= \Omega r_i \\ F_i^\theta &= 0 \end{aligned} \quad (21)$$

Similar to the governing equations, the generalized differential quadrature method should be applied to the boundary conditions. As mentioned earlier, it is assumed that the nondimensional inner and outer radii of the disk are, respectively,  $a$  and  $b$ . The following boundary conditions are used for inner and outer radii of the disk

$$\begin{aligned} r = a : u &= 0, & \theta &= \theta_{in} - (1 + 100t)e^{-100t} \\ r = b : \sigma_{rr} &= 0, & \frac{\partial \theta}{\partial r} &= 0 \end{aligned} \quad (22)$$

The above boundary conditions express a disk subjected to thermal shock at inner surface, while outer surface is thermally insulated. Furthermore, the outer surface is free of radial stress and the inner surface is free of radial deformation. Upon application of differential quadrature method, Eq. (22) takes the form

$$\begin{aligned} r = r_1 : \sum_{j=1}^N C_{1j}^{(0)} u_j &= 0, \\ \sum_{j=1}^N C_{0j}^{(1)} \theta_j &= \theta_{in} - (1 + 100t)e^{-100t} \\ r = r_N : \sum_{j=1}^N C_{Nj}^{(1)} \theta_j &= 0, \\ \sum_{j=1}^N \left( C_{Nj}^{(1)} + \frac{\bar{\lambda}}{r_N(\bar{\lambda} + 2\mu)} C_{Nj}^{(0)} \right) u_j - \sum_{j=1}^N C_{Nj}^{(0)} \theta_j &= 0 \end{aligned} \quad (23)$$

Various methods are available to apply the boundary conditions to the discreted equations of motion and energy. In this study, boundary conditions (23) are applied directly to Eq. (20). After that, the motion and energy equations may be written as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} \quad (24)$$

To complete the approximation, one should approximate the time derivatives in Eq. (24). Here, the Newmark direct integration scheme based on the constant average acceleration method ( $\alpha_N = 0.5, \beta_N = 0.25$ ) is employed [8]. Implementation of the Newmark method to Eq. (24) yields

$$\widehat{\mathbf{K}}\mathbf{X}_{j+1} = \widehat{\mathbf{F}}_{j,j+1} \quad (25)$$

where

$$\begin{aligned} \widehat{\mathbf{K}}_{j+1} &= \mathbf{K}_{j+1} + a_3 \mathbf{M}_{j+1} + a_6 \mathbf{C}_{j+1} \\ \widehat{\mathbf{F}}_{j,j+1} &= \mathbf{F}_{j+1} + \mathbf{M}_{j+1} (a_3 \dot{\mathbf{X}}_j + a_4 \ddot{\mathbf{X}}_j) + \\ &\quad \mathbf{C}_{j+1} (a_6 \dot{\mathbf{X}}_j + a_7 \ddot{\mathbf{X}}_j + a_8 \ddot{\mathbf{X}}_j) \end{aligned} \quad (26)$$

with

$$\begin{aligned} a_1 &= \alpha_N \Delta t, & a_2 &= (1 - \alpha_N) \Delta t \\ a_3 &= \frac{1}{\beta_N \Delta t^2}, & a_4 &= \frac{1}{\beta_N \Delta t}, & a_5 &= \frac{1 - 2\beta_N}{2\beta_N} \\ a_6 &= \frac{\alpha_N}{\beta_N \Delta t}, & a_7 &= \frac{\alpha_N - \beta_N}{\beta_N}, & a_8 &= \frac{\alpha_N - 2\beta_N}{2\beta_N} \Delta t \end{aligned} \quad (27)$$

Once the solution  $\mathbf{X}$  is known at  $t_{j+1} = (j+1)\Delta t$ , the first and second derivatives of  $\mathbf{X}$  at  $t_{j+1}$  can be computed from

$$\begin{aligned} \dot{\mathbf{X}}_{j+1} &= a_3 (\mathbf{X}_{j+1} - \mathbf{X}_j) - a_4 \dot{\mathbf{X}}_j - a_5 \ddot{\mathbf{X}}_j \\ \ddot{\mathbf{X}}_{j+1} &= \ddot{\mathbf{X}}_j + a_2 \dot{\mathbf{X}}_j + a_1 \ddot{\mathbf{X}}_{j+1} \end{aligned} \quad (28)$$

The resulting equations are solved at each time step using the information known from the preceding time step solution. At time  $t = 0$ , the initial values of  $\mathbf{X}$ ,  $\dot{\mathbf{X}}$ , and  $\ddot{\mathbf{X}}$  are known or obtained by solving Eq. (24) at time  $t = 0$  and are used to initiate the time marching procedure. Since the disk is initially at rest, the initial values of  $\dot{\mathbf{X}}$  and  $\ddot{\mathbf{X}}$  are assumed to be zero. In other words, the initial conditions to begin the time marching are

$$u(r, 0) = \dot{u}(r, 0) = \theta(r, 0) = \dot{\theta}(r, 0) = 0 \quad (29)$$

## VI. RESULTS AND DISCUSSION

A rotating disk made of a homogeneous/isotropic material whose thermomechanical properties are as  $\lambda = 40.4GPa$ ,  $\mu = 27GPa$ ,  $\alpha = 23 \times 10^{-6} 1/K$ ,  $\rho = 2707 kg/m^3$ ,  $K = 204 W/m$  and  $c_e = 903J/kg$  is considered. The dimensionless inside and outside radii of the disk are  $a = 1$  and  $b = 2$ . The case of generalized thermoelasticity of a stationary disk is solved previously by Bagri and Eslami [2]. In the analysis of Bagri and Eslami [2], Laplace transformation in time domain accompanied with the finite element method in radial domain is used to obtain the temporal evolution of displacement and temperature. In the next, at first two comparison studies are presented. Afterwards, numerical results of this study for the case of generalized thermoelasticity of a rotating disk are given.

## A. Comparison Study

To assure the validity of the present formulation, two comparisons are done. One for the case of stationary disk and the other for rotating disk. For the first comparison study, consider a disk with the physical and geometrical properties defined in the previous section. Mechanical and thermal boundary conditions are taken from Bagri and Eslami [2] as

$$\begin{aligned} r = a : u = 0, & \quad q = q_{in} \\ r = b : \sigma_{rr} = 0, & \quad \theta = 0 \end{aligned} \quad (30)$$

Intensity of the heat flux is considered as  $q_{in} = 1$ . the non-dimensional relaxation time is set equal to  $t_0 = 0.64$  and coupling coefficient is  $C = 0.1$ . Temporal evolution of radial displacement at the middle of the disk (at  $r = 1.5$ ) is evaluated and compared with the results of Bagri and Eslami [2]. Illustration is provided in Fig. 1. It is seen that excellent agreement is observed at the onset of comparison which guarantees the validity and accuracy of the present method.

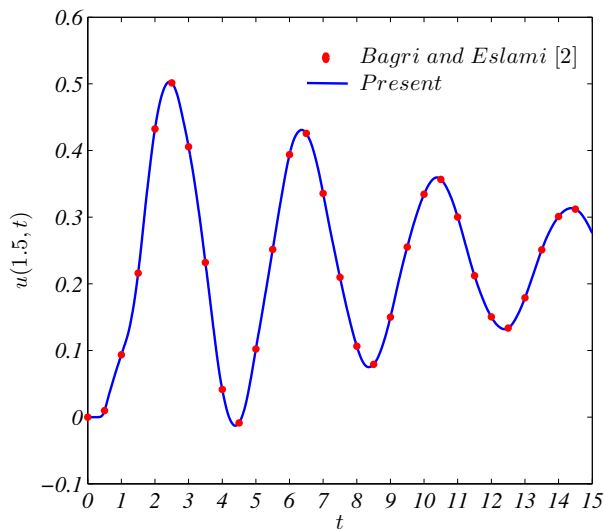


Fig. 1. Lord-Shulman theory based temporal evolution of radial displacement at  $r = 1.5$  for coupling coefficient  $C = 0.1$  and relaxation time  $t_0 = 0.64$ . For the sake of comparison, results of Fig.10 from reference [2] are read from graph.

Another comparison study is carried out in Fig. 2 for the static case of a rotating disk. A hollow disk which is

rotating with the nondimensional speed  $\Omega = 0.1$  is considered. Exact solutions for radial and centrifugal stresses are given by Hetnarski and Eslami [1]. It is seen that, results of this study match well with the analytical closed form solutions of Hetnarski and Eslami [1].

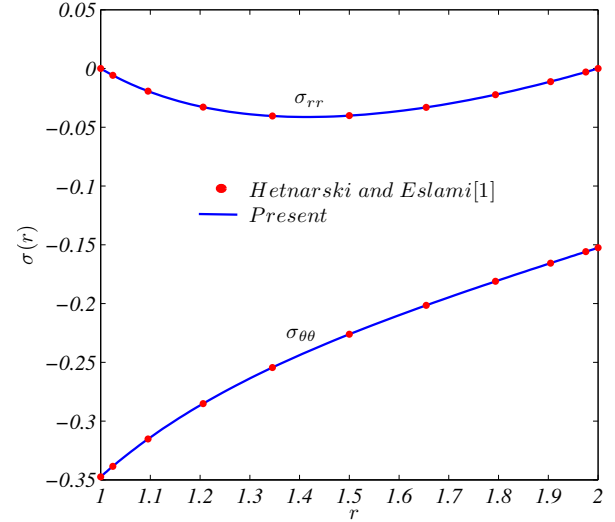


Fig. 2. A comparison on uncoupled response of a hollow disk with both inner and outer edges radial traction free subjected to the rotation parameter  $\Omega = 0.1$ .

## B. Parametric Studies

In this section, some parametric studies are conducted to examine the influences of rotation parameter on propagation of displacement, temperature, and stress waves. Thermomechanical properties and geometrical parameters are the same with those used in the previous section and boundary conditions are according to Eq. (22).

1) *Wave Front Propagation:* In this example a stationary disk is considered with relaxation time  $t_0 = 4$  and coupling parameter  $C = 0.0084567$ . This value is obtained according to Eq. (16) at reference temperature  $T_0 = 300K$ . Boundary conditions are according to Eq. (22), where the intensity of thermal shock is equal to unity. At four specific times,  $t = 0.2, 0.6, 1.2$  and  $2.2$ , propagation of thermal wave, radial displacement wave, radial stress wave, and tangential stress wave are provided respectively in Figs. 3, 4, 5, and 6. It is of worth noting that according to the dimensionless presentation (15), the speed of displacement wave is equal to unity, whereas thermal wave propagates with the speed of  $\sqrt{1/t_0} = 0.5$ . Such expectations are also observed at the figures, since after  $t = 0.2$ , thermal wave front and displacement wave front travel through, respectively,  $\Delta r = 0.1$  and  $\Delta r = 0.2$ . According to the the thermal wave speed, temperature wave front reaches the other side of the disk at  $t = 2$ , whereas at  $t = 1$  mechanical wave front is reached to the other side. Therefore, at  $t = 1.2$  mechanical wave is reflected back from the other edge of the disk. As seen from Fig. 5, the radial stress wave travels towards the outer edge compressive and reflects back tensile. This is expected since the outer edge of the disk is traction free. At  $t = 2.2$  the thermal wave is also reflected back from the

boundary. According to the figure, the temperature wave is doubled in magnitude after the reflection from the outer edge which is expected since the outer edge is thermally insulated. Again note that, when  $1 < t < 2$ , the reflected mechanical wave from the outer edge is travelling toward the inner edge. At  $t = 2$  the mechanical wave front reaches to the inner edge. At  $t = 2.2$  the mechanical wave front is reflected back from the inner edge. The tensile mechanical wave reflects back tensile since at the inner edge radial displacement is confined.

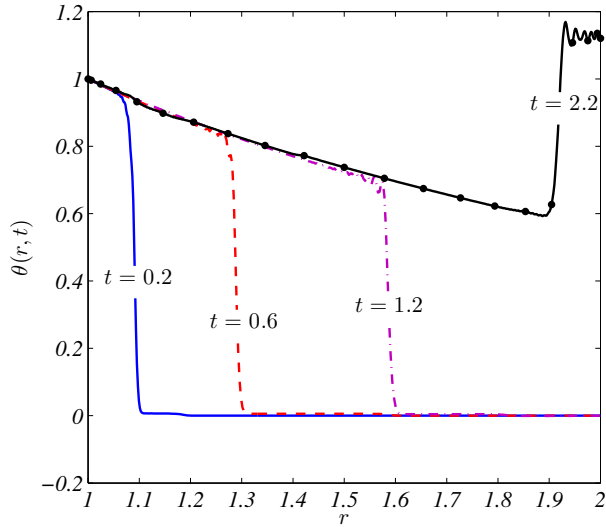


Fig. 3. Propagation of thermal wave in a stationary disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 4$  and coupling parameter  $C = 0.0084567$  at four certain times.

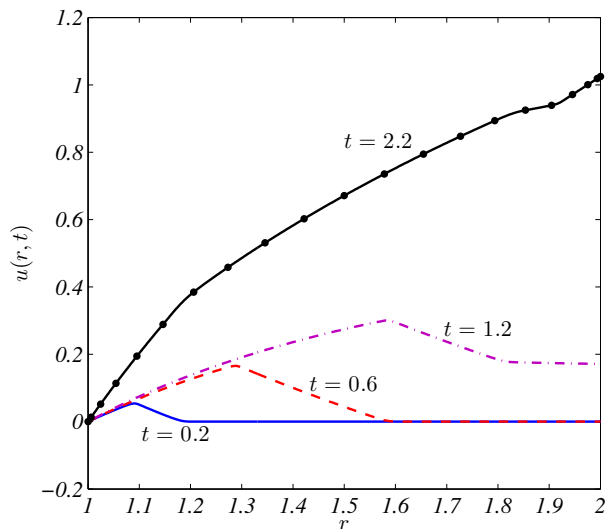


Fig. 4. Propagation of radial displacement wave in a stationary disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 4$  and coupling parameter  $C = 0.0084567$  at four certain times.

2) *Influence of rotational speed:* The next study aims to analyse the influence of rotating speed on propagation of thermal and mechanical waves. Generalized thermoelasticity response of a stationary disk is compared with two rotating

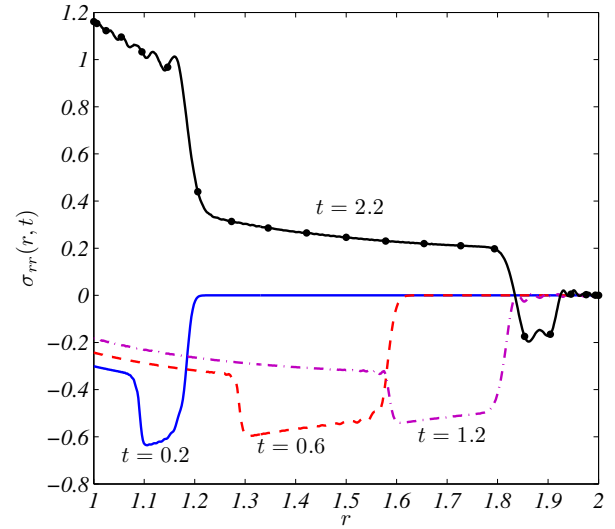


Fig. 5. Propagation of radial stress wave in a stationary disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 4$  and coupling parameter  $C = 0.0084567$  at four certain times.

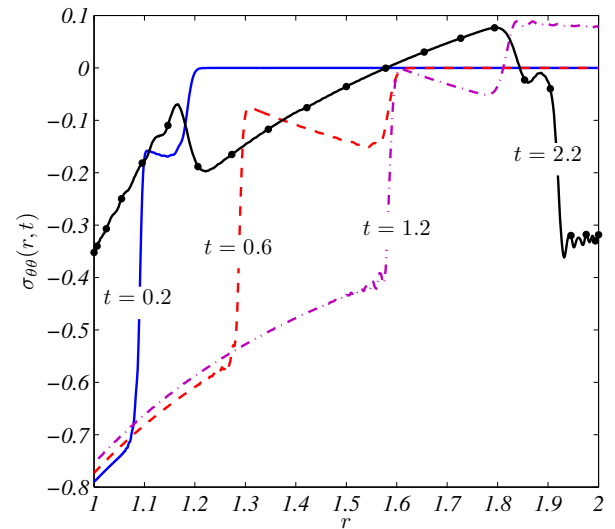


Fig. 6. Propagation of tangential stress wave in a stationary disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 4$  and coupling parameter  $C = 0.0084567$  at four certain times.

disks, with the speeds of  $\Omega = 0.5$  and  $1$ . For each case, temperature, radial displacement, radial stress, and circumferential stress waves are provided, in order, in Figs. 7, 8, 9, and 10. In this section, the non-dimensional relaxation time is set equal to  $t_0 = 1$  and coupling parameter is  $C = 0.0084567$ . From Fig. 7 it is seen that the influence of rotating speed is almost negligible on the magnitude and position of the wave front. However, as the rotating speed of the disk increases, radial displacement decreases all over the disk. The magnitude of induced compressive radial and centrifugal stresses increase in the disk. Therefore, the magnitude of stresses increase as the rotating speed of the disk increases. Note that, position of both thermal and mechanical waves are independent of the rotating speed of the disk. This is expected since the thermal and

mechanical wave speeds are functions of material properties.

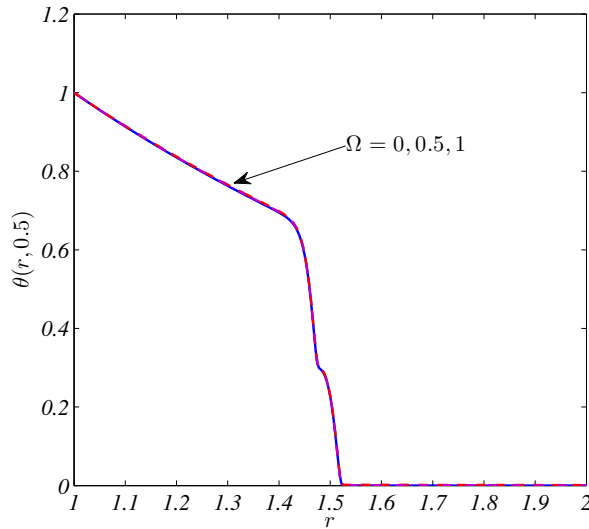


Fig. 7. Influence of rotating speed on propagation of thermal wave in a rotating disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 1$  and coupling parameter  $C = 0.0084567$  at  $t = 0.5$ .

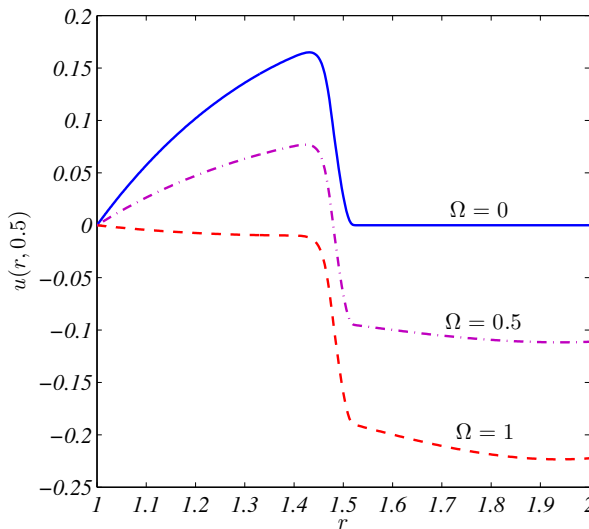


Fig. 8. Influence of rotating speed on propagation of radial displacement wave in a rotating disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 1$  and coupling parameter  $C = 0.0084567$  at  $t = 0.5$ .

## VII. CONCLUSION

An investigation is carried out on the generalized thermoelasticity response of a rotating disk made of a homogeneous and isotropic material under the single relaxation time theory of Lord and Shulman. Motion and energy equations are obtained for radial wave propagation considering the body force induced due to rotation. Equations are expressed in dimensionless presentation and discretized in radial direction according to the generalised differential quadrature. The resulting linear and coupled time dependent equations are solved using the Newmark method. Examples are given to examine the propagation

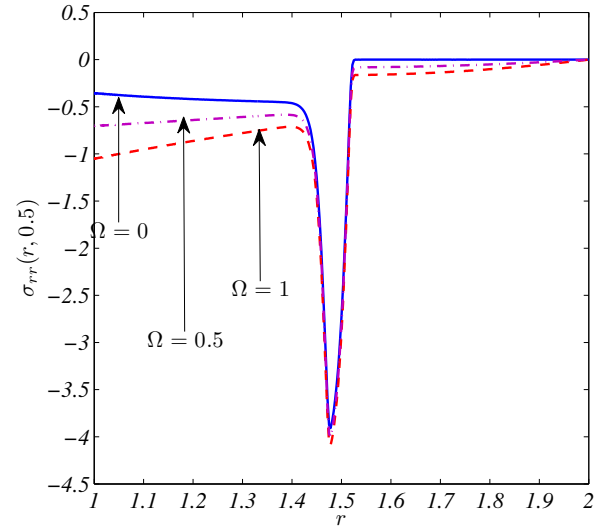


Fig. 9. Influence of rotating speed on propagation of radial stress wave in a rotating disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 1$  and coupling parameter  $C = 0.0084567$  at  $t = 0.5$ .

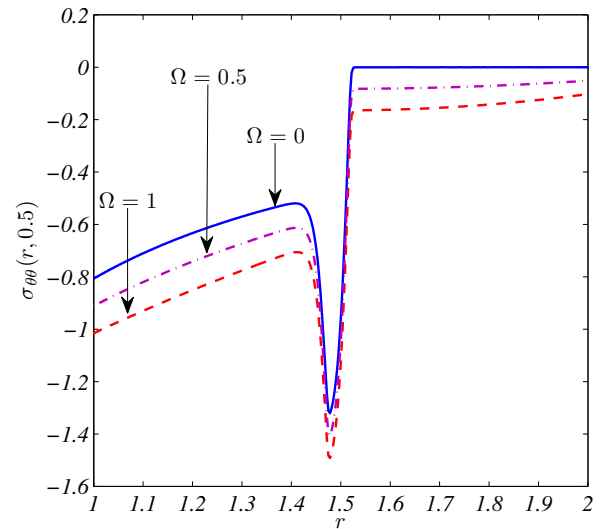


Fig. 10. Influence of rotating speed on propagation of centrifugal stress wave in a rotating disk subjected to thermal shock of magnitude  $\theta_{in} = 1$  for  $t_0 = 1$  and coupling parameter  $C = 0.0084567$  at  $t = 0.5$ .

of thermal and mechanical waves under the aforementioned conditions. Provided numerical results clearly indicates that temperature travels as a wave motion which is expected in the Lord-Shulman theory. It is shown that, temperature wave front is independent of the rotation speed, whereas the magnitude of stresses and displacement are dependent to rotating speed. As shown, rotation induces compressive forces under the studied mechanical conditions.

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